Stability of bound states in the light-front Yukawa model

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We show that in the system of two fermions interacting by scalar exchange, the solutions for $J^{\pi}=0^{+}$ bound states are stable without any cutoff regularization for coupling constant below some critical value.

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I. INTRODUCTION

A promising approach to solve the bound-state problem in field theory is the Light-Front Dynamics (LFD) [1, 2, 3, 4]. In this approach, the state vector is defined on the surface $t+z=0$. The bound states in the Yukawa model (two fermions interacting by scalar exchange) were studied in papers [5, 6] using Tamm-Dancoff method. It was found in particular that because the dominating kernel at large momenta tends to a constant, the binding energy of the $J=0^+$ state is cutoff dependent, what requires the renormalization of Hamiltonian.

The two-fermion wave functions were also considered in the explicitly covariant version of LFD [7]. The state vector is there defined on the plane given by the invariant equation $\omega\cdot x=0$ with $\omega^2=0$. The wave functions for the deuteron [8] and the pn scattering $J^{\pi}=0^+$ state [9] were found in a perturbative way and successfully applied to calculating the deuteron e.m. form factors [10] recently measured at TJNAF [11]. In papers [8, 9] the NN interaction was taken from the Bonn model which includes scalar, pseudoscalar and vector boson exchanges with cutoffs in the NN-meson vertices. The cutoff dependence of the spectrum was not analysed. LFD equations have now been solved exactly for a two fermion system in the ladder approximation with different boson exchange couplings [12].

In the present paper, we consider the solutions of the LFD equations for the scalar exchange only. We investigate the stability of the bound state solutions relative to the cutoff, disregarding the self energy contribution and renormalization. We have found a critical phenomenon for the cutoff dependence of the energy levels. The $J=0^+$ solutions are stable − not cutoff dependent when it tends to infinity − for coupling constant below some critical value. On the contrary, for coupling constants exceeding the critical value, the system is unstable (collapses) even in absence of the most singular interaction kernel in the Hamiltonian matrix. Our results are compared to those obtained in [5].

In section II we write down the equations in a convenient form for analysis. In section III, the cutoff dependence of the energy levels is studied analytically. In section IV, the results of numerical calculations are given. Section V contains the concluding remarks.

II. EQUATIONS

We start with the system of equations for the two spin components Φ_1 and Φ_2 which contain a superposition of singlet and triplet spin states:

$$\begin{aligned}
\left(M^2-M_0^2\right)\Phi_1(R_\perp,x) &= \frac{\alpha}{4\pi^2}\int\left[V_{11}(R_\perp,x;R'_\perp,x')\Phi_1(R'_\perp,x')+V_{12}(R_\perp,x;R'_\perp,x')\Phi_2(R'_\perp,x')\right]R'_\perp dR'_\perp dx', \\
\left(M^2-M_0^2\right)\Phi_2(R_\perp,x) &= \frac{\alpha}{4\pi^2}\int\left[V_{21}(R_\perp,x;R'_\perp,x')\Phi_1(R'_\perp,x')+V_{22}(R_\perp,x;R'_\perp,x')\Phi_2(R'_\perp,x')\right]R'_\perp dR'_\perp dx', \quad (1)
\end{aligned}$$

with $\alpha=\frac{g^2}{4\pi}$, $M_0^2=\frac{R_\perp^2+m^2}{x(1-x)}$ and kernels V_{ij} given by:

$$\begin{aligned}
V_{ij} &= \int_0^{2\pi} v_{ij}\left[(K^2+\mu^2)x(1-x)x'(1-x')\right]^{-1}d\phi' \\
v_{11} &= R_\perp R'_\perp(2xx'-x-x')+\left[R_\perp^2 x'(1-x')+R_\perp'^2 x(1-x)-m^2(x+x')(2-x-x')\right]\cos\phi', \\
v_{12} &= m\left[R_\perp(x+3x'-2xx'-2x'^2)-R'_\perp(x'+3x-2x'x-2x^2)\cos\phi'\right], \\
v_{21} &= m\left[R'_\perp(x'+3x-2x'x-2x^2)-R_\perp(x+3x'-2xx'-2x'^2)\cos\phi'\right],
\end{aligned}$$

$$v_{22} = R_\perp R'_\perp(2xx'-x-x')\cos\phi' + \left[R_\perp^2 x'(1-x') + R'^2_\perp x(1-x) - m^2(x+x')(2-x-x')\right]. \qquad (2)$$

The momentum K^2 reads (see eq. (7.27) from [4]):

$$K^2 = m^2\frac{x'}{x}\left(1-\frac{x}{x'}\right)^2 + \frac{x'}{x}R_\perp^2 - 2R_\perp R'_\perp\cos\phi' + \frac{x}{x'}R'^2_\perp + (x'-x)\left(\frac{m^2+R'^2_\perp}{x'(1-x')} - M^2\right) \qquad (3)$$

for $x \le x'$ and with the replacements $x \leftrightarrow x'$ $R_\perp \leftrightarrow R'_\perp$ for $x \ge x'$. Functions $\Phi_{1,2}$ are normalized as:

$$\int \left\{|\Phi_1(R_\perp,x)|^2 + |\Phi_2(R_\perp,x)|^2\right\} R_\perp dR_\perp dx = 1. \qquad (4)$$

Equations (1) and kernels (2) are taken from [5] (eqs. 3.1a-b and C1-C4) with the notations $\Phi^{1+} \equiv \Phi_1, \Phi^{2-} \equiv \Phi_2$ and $k \equiv R_\perp$, $q \equiv R'_\perp$, $y \equiv x'$.

In view of further analysis it is useful to introduce the new functions $f_{1,2}$ given by:

$$\Phi_i = \frac{1}{N(x)}\sum_j c_{ij}(R_\perp) f_j$$

$$\text{with} \qquad N(x) = \frac{2^{3/2}\pi}{\sqrt{m}}\sqrt{x(1-x)}, \qquad \hat{c} = \frac{1}{\sqrt{R_\perp^2+m^2}}\begin{pmatrix} -R_\perp & m \\ m & R_\perp \end{pmatrix}$$

and the variables k, θ ($k \ge 0$, $0 \le \theta \le \pi$) defined by:

$$R_\perp = k\sin\theta, \quad x = \frac{1}{2}\left(1 - \frac{k\cos\theta}{\sqrt{k^2+m^2}}\right). \qquad (5)$$

In this new representation, the normalization condition (4) obtains the form:

$$\frac{m}{(2\pi)^3}\int \left\{f_1^2(k,\theta) + f_2^2(k,\theta)\right\}\frac{d^3k}{\varepsilon_k} = 1, \qquad (6)$$

where $\varepsilon_k = \sqrt{m^2+k^2}$, and one gets for f_i the system of equations:

$$\begin{aligned}
\left[4(k^2+m^2) - M^2\right] f_1(k,\theta) &= -\frac{m^2}{2\pi^3}\int \left[K_{11}(k,\theta;k',\theta')f_1(k',\theta') + K_{12}(k,\theta;k',\theta')f_2(k',\theta')\right]\frac{d^3k'}{\varepsilon_{k'}}, \\
\left[4(k^2+m^2) - M^2\right] f_2(k,\theta) &= -\frac{m^2}{2\pi^3}\int \left[K_{21}(k,\theta;k',\theta')f_1(k',\theta') + K_{22}(k,\theta;k',\theta')f_2(k',\theta')\right]\frac{d^3k'}{\varepsilon_{k'}}. \qquad (7)
\end{aligned}$$

For convenience, we keep in (7) the integration over $d\phi'$, though it has been already performed in K_{ij}. The kernels K_{ij} are linearly expressed in terms of V_{ij} and their analytical expressions read:

$$\begin{aligned}
K_{ij} &= \int_0^{2\pi} \frac{\kappa_{ij}}{(K^2+\mu^2)m^2\varepsilon_k\varepsilon_{k'}}\frac{d\phi'}{2\pi}, \\
\kappa_{11} &= -\alpha\pi\left[2k^2k'^2 + 3k^2m^2 + 3k'^2m^2 + 4m^4 - 2kk'\varepsilon_k\varepsilon_{k'}\cos\theta\cos\theta' - kk'(k^2+k'^2+2m^2)\sin\theta\sin\theta'\cos\phi'\right], \\
\kappa_{12} &= -\alpha\pi m(k^2-k'^2)\left(k'\sin\theta' + k\sin\theta\cos\phi'\right), \\
\kappa_{21} &= -\alpha\pi m(k'^2-k^2)\left(k\sin\theta + k'\sin\theta'\cos\phi'\right), \qquad (8) \\
\kappa_{22} &= -\alpha\pi\left[\left(2k^2k'^2 + 3k^2m^2 + 3k'^2m^2 + 4m^4 - 2kk'\varepsilon_k\varepsilon_{k'}\cos\theta\cos\theta'\right)\cos\phi' - kk'(k^2+k'^2+2m^2)\sin\theta\sin\theta'\right],
\end{aligned}$$

and (see eq. (3.60) from [4]):

$$K^2 = k^2 + k'^2 - 2kk'\left(1 + \frac{(\varepsilon_k - \varepsilon_{k'})^2}{2\varepsilon_k\varepsilon_{k'}}\right)\cos\theta\cos\theta' - 2kk'\sin\theta\sin\theta'\cos\phi' + \left(\varepsilon_k^2 + \varepsilon_{k'}^2 - \frac{1}{2}M^2\right)\left|\frac{k\cos\theta}{\varepsilon_k} - \frac{k'\cos\theta'}{\varepsilon_{k'}}\right|.$$

In the variables k, θ, the kinetic energy in (7) is quadratic on k, the kernels are smooth in θ, and the stability of the binding energy is related to the kernels behavior at large k.

In the explicitly covariant version of LFD, the states are labeled by the eigenvalues J corresponding to the appropriate angular momentum operator [4] $\vec{J} = \vec{J}_0 + \vec{J}_{\vec{n}}$, containing, besides the free operator $\vec{J}_0$, the term $\vec{J}_{\vec{n}} = -i[\vec{n} \times \partial \vec{n}]$, where in c.m.-system $\vec{n} = \vec{\omega}/|\vec{\omega}|$. The two fermion wave function with $J = 0$ is determined by two spin components [9]. The $J = 1$ wave function is determined by six components [8] and the equations are split in two subsystems, distinguished by the eigenvalues $a = 0, 1$ of $\hat{A} = (\vec{J}\cdot\vec{n})^2$. The $J = 1, a = 0$ subsystem includes two components and the $J = 1, a = 1$ one includes four. The coupled equations obtained in this way (two for $J = 0$ and two+four for $J = 1$) correspond to the 2+2+4 systems from [5]. In this classification, the equations (7) with the kernels (8) are written for the $J = 0$ state, corresponding to $(1+, 2-)$ from [5]. In the explicitly covariant LFD these equations are directly obtained in the form (7). Their direct derivation, together with the $J = 1$ case, will be given in a more detailed publication [12]. The relation between Φ_i and f_i and the corresponding relation between K_{ij} and V_{ij} result from the different representations of the spinors used in [5] and [4]. Below we present also the results for the $J = 1, a = 0$ state, whose components are related by a linear combination to Φ^{1-}, Φ^{2+} defined in [5]. To distinguish the states, when necessary, we will attach to the kernels the index $J = 0$ or $J = 1$ and omit $a = 0$.

We would like to emphasize that the equations (1) solved in [5] are related to our explicitly covariant LFD equations (7) only by a linear tranformation of the components and a variable change. Both equations are thus strictly equivalent.

III. THE CUTOFF DEPENDENCE OF THE BINDING ENERGY

We consider the equations on the finite interval $0 < k < k_{max}$. The dependence of the solution on the cutoff k_{max} in the limit $k_{max} \to \infty$ is determined by the kernels asymptotics. Let us first analyze the kernel K_{11}. In the (k, k')-plane, when both $k, k' \to \infty$ with a fixed ratio $k'/k = \gamma$, this kernel tends to a constant. From the expressions (8) we find the asymptotics:

$$K_{11} = -\frac{2\pi^2}{m} \begin{cases} \sqrt{\gamma} A_{11}(\theta, \theta', \gamma), & \text{if } \gamma < 1 \\ \dfrac{A_{11}(\theta, \theta', 1/\gamma)}{\sqrt{\gamma}}, & \text{if } \gamma > 1 \end{cases} \tag{9}$$

with

$$A_{11}(\theta, \theta', \gamma) = \frac{\alpha'}{\sqrt{\gamma}} \int_0^{2\pi} \frac{d\phi}{2\pi} \frac{2\gamma(1 - \cos\theta\cos\theta') - (1 + \gamma^2)\sin\theta\sin\theta'\cos\phi}{(1 + \gamma^2)(1 + |\cos\theta - cos\theta'| - \cos\theta\cos\theta') - 2\gamma\sin\theta\sin\theta'\cos\phi}, \tag{10}$$

and $\alpha' = \alpha/(2m\pi)$. For convenience, we extracted the factor $\sqrt{\gamma}$ in eq. (9). When k' is fixed and $k \to \infty$, K_{11} decreases like $1/k$, and analogously for k fixed, $k' \to \infty$.

We will compare the kernels in eq. (7) with the kernel corresponding to the non-relativistic potential $U(r) = -\alpha'/r^2$. To get the exact correspondence, we take the S-wave Schrödinger equation for the wave function $\psi(k)$, with the kernel $\tilde{U}(k, k')$, Fourier transform of $U(r) = -\alpha'/r^2$, and we substitute there $\psi(k) = f(k)\sqrt{m/\varepsilon_k}$. The equation obtained for $f(k)$ has the relativistic form (7). It contains the factor $1/\varepsilon_{k'}$ in the integration volume and the kernel $K_{-\alpha'/r^2}(k, k') = \tilde{U}(k, k')\sqrt{\varepsilon_k \varepsilon_{k'}}/m$. For $k'/k = \gamma$ fixed the latter has the asymptotics (9) with the constant α' instead of the function A_{11}. For k' fixed, $k \to \infty$ it decreases like $1/\sqrt{k}$, and analogously for k fixed, $k' \to \infty$.

The kernel K_{22}, when $k \to \infty$ and $k'/k = \gamma$ fixed, has also the asymptotics (9), however with an unbounded function A_{22}, in contrast to A_{11} (see below). When $k \to \infty$ and k' fixed (and vice versa) it tends to a constant. Therefore it always dominates over K_{11} and over $K_{-\alpha'/r^2}(k, k')$.

To disentangle the two different sources of collapse, we first consider the one channel problem for the component f_1 with the kernel K_{11}, and we remove the second equation from (7). We analyse the domain of $k'/k = \gamma$ fixed, where K_{11} has its maximal asymptotics values. The θ, θ' domain is finite and the function $A_{11}(\theta, \theta', \gamma)$ has no singularities in θ, θ'. Therefore we majorate it by its maximal value. If the stronger, majorated kernel results in stable bound states, the exact one results in stability too. This method to analyse the cutoff dependence is equivalent to applying the sufficient condition of stability proposed in [13]. The inspection of (10) shows that for fixed γ, the maximum of A_{11} is achieved at $\theta = \theta'$, where it has the form: $A_{11}(\theta = \theta', \gamma) = \alpha'\sqrt{\gamma}$ independent of θ. Note that $A_{22}(\theta = \theta', \gamma) = \alpha'/\sqrt{\gamma}$ is unbounded when $\gamma \to 0$. Then we majorate the function $A_{11}(\theta = \theta', \gamma)$ by its maximal value relative to γ: $A_{11}^{max} = \alpha'$, which is evidently achieved at $\gamma = 1$. With this value of A_{11}, the kernel (9) exactly coincides with the kernel $K_{-\alpha'/r^2}(k, k')$. As well known [14], with the potential $-\alpha'/r^2$, the binding energy does not depend on cutoff if $\alpha' < 1/(4m)$, what restricts the coupling constant to $\alpha < \pi/2$. If $\alpha' > 1/(4m)$, the system collapses, what is manifested by the fact that the binding energy tends to $-\infty$ when $k_{max} \to \infty$. In this system, there exists a critical value α_c of the coupling constant, below which the binding energy is stable. Majorating the kernel, we underestimate α_c. We calculated also this value [12], not majorating the function $A_{11} = \alpha'\sqrt{\gamma}$, but taking into

account its dependence on γ. In this way we find $\alpha_c = \pi$, instead of $\pi/2$, when $\sqrt{\gamma}$ was replaced by 1. Because of majorating the kernel in the variables θ, θ', this value should be still smaller than the true α_c but it is in the range $3 < \alpha_c < 4$ we found numerically (see next section).

In the two-channel problem, the kernel dominating in asymptotics is K_{22}. In the case $J = 0$ it is positive and corresponds to repulsion. Because of that, this channel does not lead to any collapse. This repulsion cannot prevent from the collapse in the first channel (for enough large α), since due to coupling between two channels the singular potential in the channel 1 "pumps out" the wave function from the channel 2 into the channel 1. So, in the coupled equations system (7) the situation with the cutoff dependence is the same as for one channel.

Let us now consider the state $J = 1, a = 0$. The asymptotics of the kernel $K_{22}^{(J=1)}$ is the same than $-K_{22}^{(J=0)}$, it is negative and corresponds to attraction. Since it always dominates over $K_{-\alpha'/r^2}(k, k')$, this attraction is stronger than in the $-\alpha'/r^2$ potential. Therefore it results in a collapse for any value of the coupling constant. In the paper [5], this situation corresponds to the two-fermion state described by the components Φ^{1-}, Φ^{2+}.

IV. NUMERICAL RESULTS

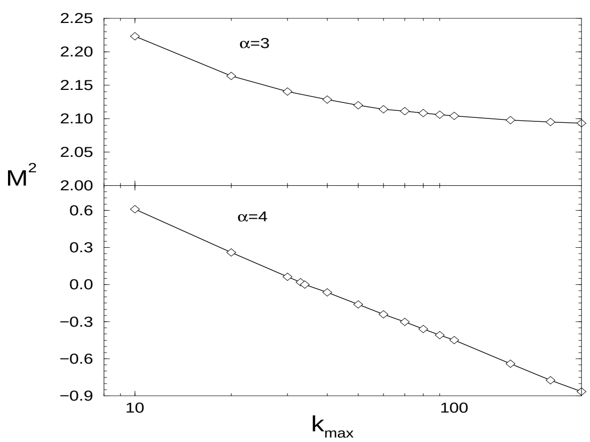

FIG. 1: Cutoff dependence of the binding energy in the $J = 0$ or $(1+, 2-)$ state, in the one-channel problem (f_1), for two fixed values of the coupling constant below and above the critical value.

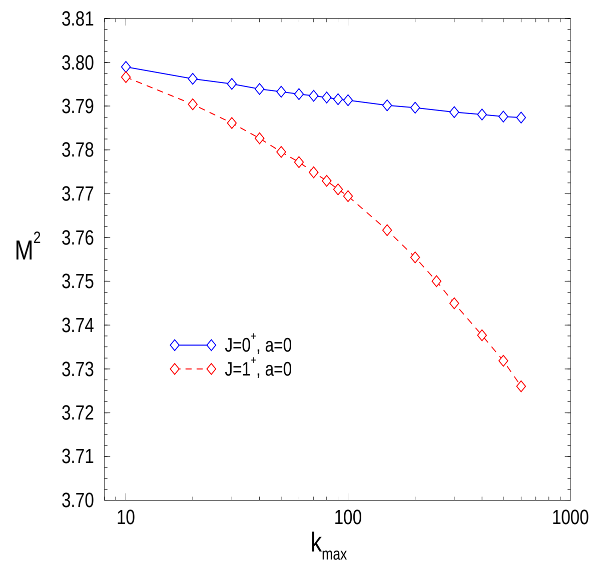

FIG. 2: Cutoff dependence of the binding energy, for $J = 0$ $(1+, 2-)$ and $J = 1, a = 0$ $(1-, 2+)$ states, in the two-channel problem ($\alpha = 1.184$).

The preceding results are confirmed by numerical calculations. The constituent masses were taken equal to m=1 and the mass of the exchanged scalar μ=0.25.

We first present the results given by the single equation for f_1 with kernel K_{11} in the $J = 0$ case. We have plotted in figure 1 the mass square M^2 of the two fermion system as a function of the cutoff k_{max} for two fixed values of the coupling constant below and above the critical value α_c. In our calculations, the cutoff appears directly as the maximum value k_{max} up to which the integrals in (7) are performed. One can see two dramatically different behaviors depending on the value of the coupling constant α. For $\alpha = 3$, i.e. $\alpha < \alpha_c$, the result is convergent. For $\alpha = 4$, i.e. $\alpha > \alpha_c$, the result is clearly divergent. M^2 decreases logarithmically as a function of k_{max} and becomes even negative. We would like to notice that this divergence is not associated with the non decreasing behavior of the K_{22} kernel but with the existence of a critical value of the coupling constant separating two dynamical regimes. This property is due only to the large k behavior of K_{11}. Though the negative values of M^2 are physically meaningless, they are formally allowed by the equations (1) and (7). The first degree of M does not enter neither in the equation nor in the kernel, and M^2 crosses zero without any singularity. The value of α_c does not depend on the exchange mass μ. For $\mu \ll m$, e.g. $\mu \approx 0.25$, its existence is not relevant in describing physical states since any solution with positive M^2, stable relative to cutoff, corresponds to $\alpha < \alpha_c$. For $\mu \sim m$ one can reach the critical α for positive, though small values of M^2.

We consider now the full Yukawa problem as given by the two coupled equations (7). In figure 2 are displayed the variations of M^2 for $J=0$ or $(1+,2-)$ and $J=1, a=0$ or $(1-,2+)$ states as a function of the cutoff k_{max}. The value of the coupling constant for both J is $\alpha=1.184$, the same that in Fig. 2 of [5], below the critical value. Our numerical values are in agreement with the results for the cutoff $\Lambda \leq 100$ presented in this figure [5], but our calculation at larger k_{max} leads to different conclusion for the $J=0$ state. We first notice a qualitatively different behavior of the two states. In what concerns $J=0$, the curve becomes flat when k_{max} increases, – with a 0.1% variation in M^2 when changing k_{max} from 50 to 600. We thus conclude to the stability of the state with $J=0$, as expected from our analysis in sect. III.

On the contrary, for $J=1, a=0$ the value of $M^2(k_{max})$ decreases faster than logarithmically what indicates – as found in [5] – a collapse. As mentioned above, the asymptotics of the $K_{22}^{(J=1)}$ kernel is the same as the $K_{22}^{(J=0)}$ one but with an opposite sign, i.e. it is attractive, what leads to unstability for any value of α. We found the same result when solving the $J=0$ equations with the opposite sign of $K_{22}^{(J=0)}$.

V. CONCLUSION

The Light-Front solutions of the two fermion system interacting via a scalar exchange have been obtained. We have found that the $J=0$ – or $(1+,2-)$ – state is stable (i.e. convergent relative to the cutoff $k_{max} \to \infty$) for coupling constant below some critical value, in a way similar to what is known in non relativistic quantum mechanics for the $-\alpha'/r^2$ potential. In this point, our conclusion differs from the one settled in [5], where it was stated that the integrals in eqs. (1) diverge logarithmically with cutoff. Above the critical value the system collapses, what manifests as an unbounded cutoff dependence of M^2. In the $J=1, a=0$ – or $(1-,2+)$ – state the system is found to be always unstable, in agreement with [5].

The origin of this unstability for the $J=0$ state differs from $J=1$. We have found that the $K_{22}^{(J=0)}$ dominating kernel does not generate a collapse because it is repulsive. The unstability in this case is related to K_{11}.

These results should be taken into account when carrying out the renormalization procedure. The explicitly covariant LFD may be efficient for solving this problem, like it has proved to be fruitful for analyzing the Yukawa model.

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